Universality in modelling pattern forming of polariton condensates

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Outline

- Introduction: universality of OPEs
- Lasers – polariton condensates – atomic condensates
- Maxwell-Bloch equations for a laser
- Ginsburg-Landau vs Swift-Hohenberg equations
- Pattern formation and stability
    - Homogeneous OPE
    - Inhomogeneous pumping
    - Inhomogeneous energy (trapping)
    - Vortex lattices

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N.G.Berloff and J.Keeling "Universality in modelling non-equilibrium polariton condensates", chapter in the book "Quantum fluids:hot topics and new trends" ed. A. Bramati and M. Modugno, Springer-Verlag (2012).

M.Borgh, G.Franchetti, J.Keeling and N.G. Berloff in preparation PRA, (2012)

Universality of OPEs

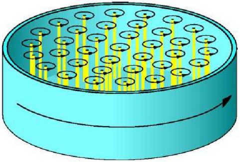
Order Parameter Equations (OPEs) describe:

- relaxation toward an equilibrium configuration $\partial_t \psi = -\Gamma \partial_\psi \mathcal{F}$;
- phase evolution in a conservative system (Hamiltonian dynamics);
- mixture of the two.

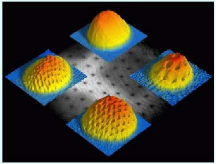
The structure of the energy functional $\mathcal{F}$ is determined by the symmetries of order parameter space, e.g. Ginsburg-Landau energy functional:

$$\mathcal{F} = \int dV \nabla\psi \cdot \nabla\psi^* + (\mu - U_0|\psi|^2)^2$$

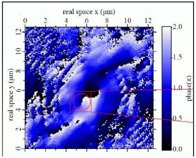
Hydrodynamic interpretation of OPEs: vortices

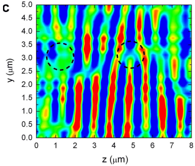


^{4}He Vinen (1956), ^{3}He-B Helsinki group 80s



Rb^{87} Wolfgang Ketterle group (2001)

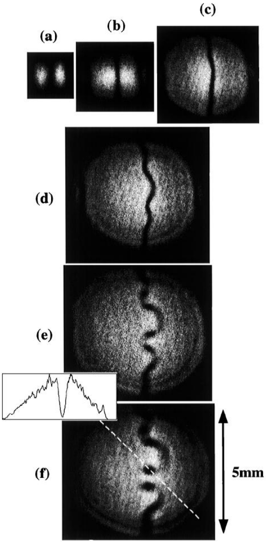




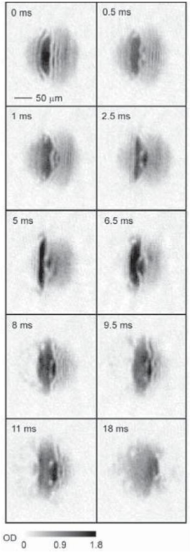
Lagoudakis et al. Nature Phys (2008); Nowik-Boltyk et al, Nature Com. (2012)

Universality: dark Soliton and "snake" instability

In nonlinear optics



In atomic BECs



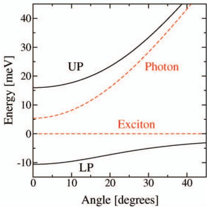
Lasers: Maxwell-Bloch equations

Laser dynamics is described by coupling Maxwell equations with Shrödinger equations for N atoms confined in the cavity.
MBE—E in cavity modes coupled to collective variables that describe the polarisation and population of the gain medium.
Lasers are classified depending on the relative order of the loss rates for the electric field, compared to the decay rates of the gain medium polarisation and population.
Two homogeneous solutions: $\psi = 0$ and $\psi = const$

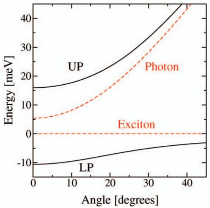
Instabilities

population	nonlasing	lasing
Fast	cSH	cSH + KS
Slow	cSH + population mean flow	cSH + KS + mean flow

Polariton condensates



Emission follows the bare photon dispersion $\rightarrow$ regular lasing;
Emission follows the lower polariton dispersion $\rightarrow$ polariton condensation;
Small pumping and losses $\rightarrow$ equilibrium Bose-Einstein condensates.
Unified approach to describe the transition from normal lasers to the equilibrium BECs via polariton condensates! Photon lasers: (1) resonator for the electromagnetic field; (2) gain medium; (3) excitation mechanism for the gain medium.
Polariton condensates: stimulated scattering within the set of polariton modes.
Idea: *Given the universality of OPEs, write a single OPE which captures these different regimes by varying appropriate parameters*



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Maxwell-Bloch equations for a laser

$$\frac{\partial E}{\partial t} - i\nabla^2 E = P_g - P_a - (1 + i\Delta_e)E,$$
$$\tau_{\perp g}\frac{\partial P_g}{\partial t} + (1 + i\Delta_g)P_g = EG,$$
$$\tau_{\perp a}\frac{\partial P_a}{\partial t} + (1 + i\Delta_a)P_a = EA,$$
$$\tau_g\frac{\partial G}{\partial t} = G_0 - G - \frac{1}{2}(E^* P_g + EP_g^*),$$
$$\tau_a\frac{\partial A}{\partial t} = A_0 - A - \frac{D}{2}(E^* P_a + EP_a^*),$$

E is the envelope of the electric field,
G and A are the population differences for gain and absorption media,
P_g and P_a are the envelopes of polarisation for gain and absorbtion media;
G_0 and A_0 are the stationary values of the population difference;
$D = \tau_{\perp a}\tau_a\mu_a^2/(\tau_{\perp g}\tau_g\mu_g^2)$ is the relative saturability of gain and loss media;
$\tau_{\perp a,g}$ and $\tau_{a,g}$ are the relaxation times for atomic polarisations and population differences scaled by the cavity relaxation time.

Fast reservoir dephasing limit $\tau_{\perp a}, \tau_{\perp g} \ll 1$

$$
\begin{aligned}
\frac{\partial E}{\partial t} - i\nabla^2 E &= P_g - P_a - (1 + i\Delta_e)E, \\
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$$P_g = \frac{EG}{1 + i\Delta_g} \qquad ,$$

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$$P_g = \frac{EG}{1 + i\Delta_g} - \tau_{\perp g}\frac{(EG)_t}{(1 + i\Delta_g)^2},$$

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$$P_g = \frac{EG}{1 + i\Delta_g} - \tau_{\perp g}\frac{(EG)_t}{(1 + i\Delta_g)^2},$$

$$(1 + i\eta)\frac{\partial e}{\partial t} - i(\nabla^2 - \Delta_e)e = [(1 - i\Delta_g)g - (1 - i\Delta_a)a - 1]e,$$

where $\eta = -2\tau_{\perp g}g\Delta_g/(1 + \Delta_g^2) + 2\tau_{\perp a}a\Delta_a/(1 + \Delta_a^2)$ and rescaled $e = E/(1 + \Delta_g^2)$ etc.

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Fast reservoir population relaxation $\tau_g, \tau_a \ll 1$

$$
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&\tau_g \frac{\partial g}{\partial t} = g_0 - (1 + |e|^2)g, \\
&\tau_a \frac{\partial a}{\partial t} = a_0 - (1 + d|e|^2)a.
\end{aligned}
$$

Giving

$$
g = \frac{g_0}{1 + |e|^2}, \qquad a = \frac{a_0}{1 + d|e|^2}.
$$

Close to emission threshold $|e|^2 \ll 0$, expanding in small $|e|$ gives the complex Ginsburg-Landau equation

$$
(i - \eta)\frac{\partial e}{\partial t} = -\nabla^2 e + Ve + U|e|^2 e + i[\alpha - \beta|e|^2]e,
$$

where we let $\alpha = g_0 - a_0 - 1$, $\beta = g_0 - a_0$, $U = da_0\Delta_a - g_0\Delta_g$, $V = g_0\Delta_g - a_0\Delta_a$.

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Giving

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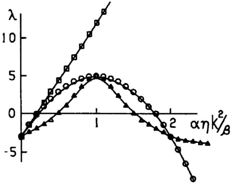
The cGL equation – no mode selection!

Perturbation growth exponent
vs perturbation wavenumber
squares–cGLE
circles – cSHE
triangles – MBE



The cGL equation does not take into account the selection of transverse modes.

The lasers emit particular transverse modes that depend on the length of the resonator.

The cGL equation – no mode selection!

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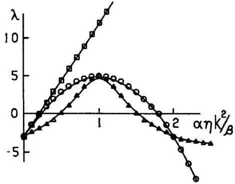

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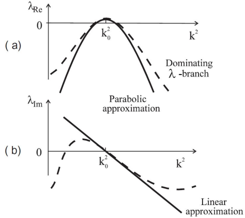
Heuristically,



Lowest degree of approximation

$$\lambda = \alpha - \delta(k^2 - k_c^2)^2 + i(k^2 - k_c^2),$$

where α is a control parameter that takes $Re(\lambda)$ into the positive range of values.

$$\frac{\partial E}{\partial t} - i(\nabla^2 - \Delta_e)E = P_g - P_a - E,$$

$$+ \text{ 4 equations on } P_g, P_a, G, A$$

Following ideas of Lega et al PRL 1994

Assume $\nabla^2 - \Delta_e$ is small ($= \epsilon(\nabla^2 - \Delta_e)$) and introduce two small time scales: $T_1 = \epsilon t,\ T_2 = \epsilon^2 t$, so that $\partial_t = \epsilon \partial_{T_1} + \epsilon^2 \partial_{T_2}$.

Steps:

(1) Write functions as asymptotic expansions in ϵ: $E = \sum \epsilon^n E_n$ etc. Equations have form $\mathcal{L}E_n = g_n$.

(2) Fredholm Alternative: require g_n to be orthogonal to the solutions of the adjoint homogeneous problem $\mathcal{L}^* E_n = 0$.

(3) Finally, at a given order, obtain a closed equation for the evolution of one single variable — order parameter ($\psi = E$)

The complext Swift-Hohenberg equation

$$(1+i\eta)\frac{\partial\psi}{\partial t} = i(\nabla^2-\Delta_e)\psi - \delta(\nabla^2-\Delta_e)^2\psi + (\alpha - iV)\psi - (\beta + iU)|\psi|^2\psi$$

Energy relaxation $\eta = -2G_0\Delta_g\tau_{\perp g} + 2A_0\Delta_a\tau_{\perp a}$,
Coefficient of superdiffusion $\delta = \tau_{\perp g}^2 G_0 - \tau_{\perp a}^2 A_0$,
Effective pumping $\alpha = G_0 - A_0 - 1$,
Effective repulsive potential $V = G_0\Delta_g - A_0\Delta_a$,
Cubic damping $\beta = G_0 - A_0 D$,
Interaction potential $U = A_0 D\Delta_a - G_0\Delta_g$.

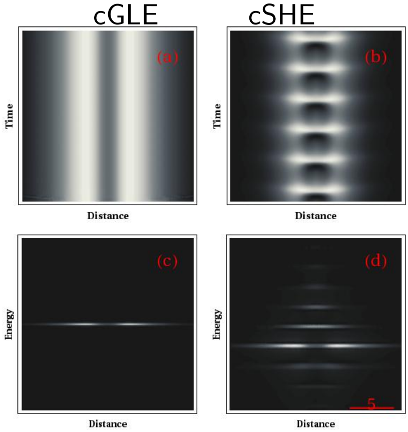
Slow population evolution: $\tau_{\perp g,a}/\tau_{g,a} \ll 1$

$$\begin{aligned}
(1+i\eta)\frac{\partial\psi}{\partial t} &= i(\nabla^2-\Delta_e)\psi - \delta(\nabla^2-\Delta_e)^2\psi + (G - A - 1)\psi \\
&- i(\Delta_g G - \Delta_a A)\psi, \\
&\tau_g\frac{\partial G}{\partial t} = G_0 - (1+|\psi|^2)G, \\
&\tau_a\frac{\partial A}{\partial t} = A_0 - (1+D|\psi|^2)A.
\end{aligned}$$

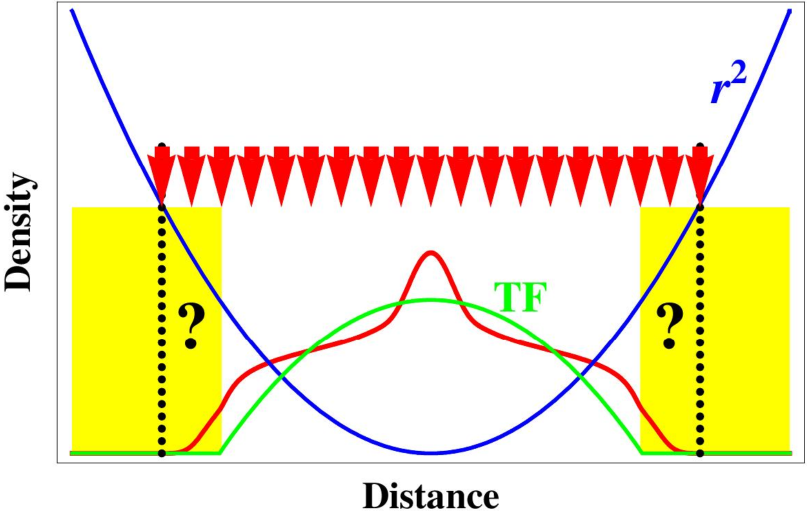
Inhomogeneous pumping

$$(1 + i\eta(P))\frac{\partial\psi}{\partial t} = \left(P(\mathbf{r}) - \gamma_c - \lambda P(\mathbf{r})|\psi|^2\right)\psi + i(\nabla^2 - V(P) - |\psi|^2)\psi$$
$$+2\delta\Delta_e\nabla^2\psi - \delta\nabla^4\psi, .$$

We take $\delta = 0.1$, $\Delta_e = -0.1$



Inhomogeneous energy (trapping)



Stability analysis

Neglect quantum pressure terms, superdiffusion and re-scale:

$$\partial_t \rho + \nabla \cdot (\rho \nabla \phi) = \left(\tilde{\alpha} - \tilde{\beta}\rho + 2\tilde{\eta}\partial_t \phi - 2\tilde{\delta}(\nabla\phi)^2 \right) \rho,$$
$$2\partial_t \phi + (\nabla\phi)^2 + r^2 + \rho = \tilde{\delta}(2\tilde{\Delta}_e \nabla^2 - \nabla^4)\phi.$$

Without dissipative terms linearise using $\rho \to \rho + h e^{-i\omega t}, \phi \to \phi + \varphi e^{-i\omega t}$ to get [Stringary PRL 1998] normal modes with frequencies $\omega_{ns} = \sqrt{2n^2 + 2(s+1)n + s}$ and density profiles given by hypergeometric functions $h(r,\theta) \propto {}_2F_1(-n, n+s+1; s+1, r^2) e^{is\theta} r^s$; here n is a radial quantum number, and s is an angular quantum number.

The first order correction

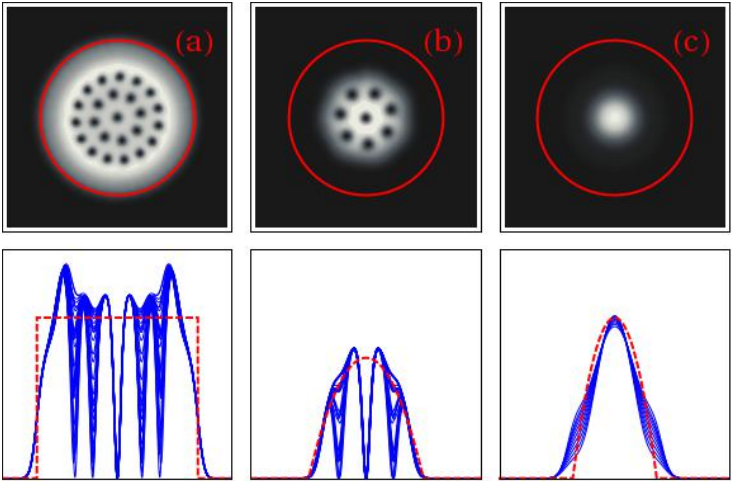
$$\omega_{ns}^{(1)} = \frac{i}{2N} \int 2\pi r dr [(h_{ns}^{(0)})^2 \left(\tilde{\alpha} - \tilde{\eta}\mu - (2\tilde{\beta} + \tilde{\eta})\mu \right)$$
$$+ \tilde{\delta} h_{ns}^{(0)} \left(\tilde{\Delta}_e - \frac{1}{2}\nabla^2 \right) \nabla^2 h_{ns}^{(0)}]$$

which at large s $\omega_{ns}^{(1)} \to i\tilde{\beta}\tilde{\alpha}/(2\tilde{\beta} + 3\tilde{\eta}) > 0$

Instability!

Vortex Lattices

cGLE with $\eta = 0$ | **cGLE with $\eta = 0.2$** | **cSHE $\delta = 0.2, \Delta_e = -0.2$**



Red dashed lines — analytical solutions

Summary

- Connection between lasers, polariton condensates and equilibrium condensates from the common framework based on the MBE.
- The complex Swift-Hohenberg equation should be applicable to polariton condensates.
- The pattern formation in the framework of the cSH equations have been well-studied for lasers.
- Some of these phenomena may be achieved in polariton condensates.
- The stronger nonlinearities and different external potentials (engineered or due to disorder) may lead to novel properties of the system exhibiting effects not seen in normal lasers.
- Microscopic modelling: quantum kinetic Boltzmann equation to model non-condensed particles.